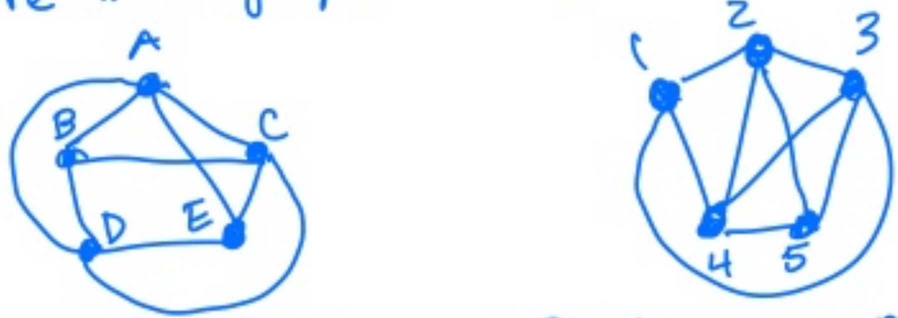
Questions

① Are these graphs isomorphic? Prove your response.



② Are these graphs isomorphic? Prove your response.

③ These are adjacency matrices of two graphs. Are the graphs isomorphic? Can you tell without drawing them?

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

One algorithm of solving the problem of whether two graphs are isomorphic:

① Write down every possible bijection F between the two sets of vertices.

② Check whether the condition

$$\{e, f\} \in E_1 \iff \{F(e), F(f)\} \in E_2$$

$e, f \in V_1$

holds for each bijection.

But: If we have N vertices, we would have to check ② $N! = N(N-1)\cdots 3\cdot 2\cdot 1$ times.

There is no algorithm known that is "polynomial time" — time required $\leq p(N)$ polynomial.

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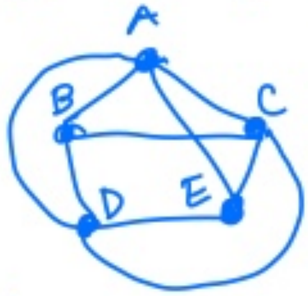
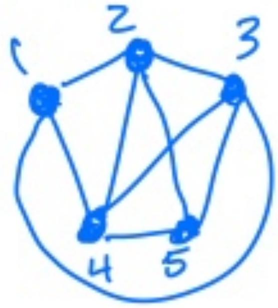


$V_1 = \{A,B,C,D,E\}$



$V_2 = \{1,2,3,4,5\}$

Yes

Proof: Let $F: V_1 \to V_2$ be defined by

$$F\begin{pmatrix}A\\B\\C\\D\\E\end{pmatrix} = \begin{pmatrix}2\\1\\3\\4\\5\end{pmatrix}$$

$\left(\text{or } \begin{array}{l} A \to 1 \\ B \to 5 \\ C \to 2 \\ D \to 3 \\ E \to 4 \end{array}\right)$

let's check

Edges of 1^{st} Graph $\{\{A,B\},\{A,C\},\{A,D\},\{A,E\},$
$\{B,C\},\{B,D\},\{C,D\},\{C,E\},\{D,E\}\}$ missing $\{B,E\}$

$\downarrow F$

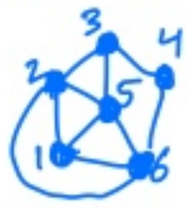
Edges of 2^{nd} graph $\{\{2,1\},\{2,3\},\{2,4\},\{2,5\}$
$\{1,3\},\{1,4\},\{3,4\},\{3,5\},\{4,5\}\}$
missing: $\{1,5\}$

$\therefore$ F is an isomorphism.

Lesson learned: ① If $F: V_1 \to V_2$ yields an isomorphism between (V_1,E_1) and (V_2,E_2),
then $\deg(v) = \deg(F(v))$ for every vertex v.

② Are these graphs isomorphic? Prove your response.

$G_1 = (V_1, E_1)$



$G_2 = (V_2, E_2)$

They are not isomorphic.

Proof: For G_2,
the vertices 2, 5, 6 have degree 4, and they are all connected to the same node (1). The vertices 1, 3 have degree 3, and vertex 4 has degree 2.

For G_1, the vertices 2, 3, 5 have degree 4, and there is no vertex that all 3 are connected to. Also, the vertices 1, 4 have degree 3, and the vertex 6 has degree 2.
Because of this, there is no graph isomorphism between the two graphs. ☒

Proof #2: The vertex of degree 2 in the first graph is connected to two vertices of degree 4. In the second graph, the vertex of degree 2 is connected to a vertex of degree 3 and a vertex of degree 4. Thus, there is no graph

Isomorphism between V_1 & V_2 for G_1 & G_2.

③ These are adjacency matrices of two graphs. Are the graphs isomorphic? Can you tell without drawing them?

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Yes. Proof: We will switch labeling of the first two vertices in the first matrix

$\Leftrightarrow$ We will switch the first two rows, then 1st & columns.

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[\substack{2 \\ \text{rows}}]{\substack{\text{switch} \\ 1^{st}}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

switch the first two columns

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \Rightarrow 2^{nd} \text{ adjacency matrix.}$$

$\therefore$ The graphs are isomorphic, since after relabeling, the adjacency matrices are the same.

Important fact: Two graphs are isomorphic

$\Longleftrightarrow$ One of the adjacency matrices can be transformed to the other one by a rearrangement (permutation) of the rows + the same permutation of the columns.